On a functional equation related to the intermediate long wave equation

2004 J. Phys. A: Math. Gen. 37 L399

(http://iopscience.iop.org/0305-4470/37/32/L02)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) L399–L406 PII: S0305-4470(04)82356-4

LETTER TO THE EDITOR

On a functional equation related to the intermediate long wave equation

A N W Hone and V S Novikov[1]

Institute of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury CT2 7NF, UK

E-mail: anwh@kent.ac.uk and vn9@kent.ac.uk

Received 18 June 2004
Published 28 July 2004
Online at stacks.iop.org/JPhysA/37/L399
doi:10.1088/0305-4470/37/32/L02

Abstract

We resolve an open problem stated by Ablowitz *et al* (1982 *J. Phys. A: Math. Gen.* **15** 781) concerning the integral operator appearing in the intermediate long wave equation. We explain how this is resolved using the perturbative symmetry approach introduced by one of us with Mikhailov. By solving a certain functional equation, we prove that the intermediate long wave equation and the Benjamin–Ono equation are the unique integrable cases within a particular class of integro-differential equations. Furthermore, we explain how the perturbative symmetry approach is naturally extended to treat equations on a periodic domain.

PACS number: 03.65.Ge

1. Introduction

The propagation of nonlinear internal waves along the interface of two fluids of different density is described by several different nonlinear equations. In the deep-fluid limit, the relevant equation is the Benjamin–Ono equation [3, 13], given by

$$u_t + 2uu_x + \mathcal{H}(u_{xx}) = 0, \tag{1}$$

where the symbol $\mathcal{H}$ denotes the Hilbert transform operator

$$\mathcal{H}(u(x)) := \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{u(y)}{y - x}\,\mathrm{d}y. \tag{2}$$

The Korteweg–de Vries (KdV) equation, on the other hand, pertains to the shallow-fluid limit:

$$u_t + 6uu_x + u_{3x} = 0. \tag{3}$$

[1] On leave from Landau Institute for Theoretical Physics, Moscow.

For the case of nonlinear waves in a fluid of finite depth Joseph [9] derived the intermediate long wave equation (ILW), which is given as

$$u_t + \delta^{-1}u_x + 2uu_x + \mathcal{T}(u_{xx}) = 0, \tag{4}$$

in terms of the coth transform

$$\mathcal{T}(u(x)) := -\frac{1}{2\delta} PV \int_{-\infty}^{\infty} \coth\left(\frac{\pi}{2\delta}(x-y)\right) u(y)\, \mathrm{d}y. \tag{5}$$

The ILW equation is intermediate between Benjamin–Ono and KdV, in the sense that the limit $\delta \to \infty$ yields (1), while $\delta \to 0$ gives (3). The remarkable fact is that all three equations (1), (3), (4) are integrable in the sense that they admit multi-soliton solutions [6] and are solvable by the inverse scattering transform (see, for example, [2]).

In the work [1] of Ablowitz *et al* the ILW equation was considered on a periodic domain rather than for the case of waves on the infinite domain decaying at infinity. The periodic case requires an alternative definition of the operator $\mathcal{T}$ compared with (5), which we discuss below. Simple periodic solutions of the ILW were treated extensively by Parker [14]. An important open problem, stated in [1], concerns the classification of all integrable equations of the form (4), where $\mathcal{T}$ is an integral operator satisfying the following conditions:

$$\mathcal{T}(u\mathcal{T}v + v\mathcal{T}u) = (\mathcal{T}u)(\mathcal{T}v) - uv, \tag{6}$$

$$\int_{-\infty}^{\infty} (u\mathcal{T}v + v\mathcal{T}u)\, \mathrm{d}x = 0. \tag{7}$$

In [1] it is stated that (6) and (7) arise as necessary conditions for equation (4) to have infinitely many conservation laws, as obtained on the infinite domain in [17]. Moreover, the authors of [1] verify that the conditions are satisfied by the coth transform operator (5) even in the periodic case, by writing $\mathcal{T}$ in terms of a Fourier series.

The purpose of this letter is to present the general solution to the problem stated by Ablowitz *et al*, as well as explaining how it can be derived in a straightforward way in the framework of the perturbative symmetry approach [11].

Let us assume that the operator $\mathcal{T}$ has the Fourier symbol $\mathrm{i}\hat{f}(k)$, so that

$$\mathcal{T}(u(x)) = \mathrm{i}\int_{-\infty}^{\infty} \hat{f}(k)\hat{u}(k)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k,$$

where the above is interpreted as a principal value (PV) integral where necessary, and the hat denotes the Fourier transform

$$\hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

Then writing the first condition (6) in terms of Fourier integrals over the whole real line, dividing by $\mathrm{i}^2 = -1$ and symmetrizing gives

$$\iint \hat{f}(k+\ell)(\hat{f}(k)+\hat{f}(\ell))\hat{u}(\ell)\hat{v}(k)\,\mathrm{e}^{\mathrm{i}(k+\ell)x}\,\mathrm{d}k\,\mathrm{d}\ell = \iint (\hat{f}(k)\hat{f}(\ell)+1)\hat{u}(\ell)\hat{v}(k)\,\mathrm{e}^{\mathrm{i}(k+\ell)x}\,\mathrm{d}k\,\mathrm{d}\ell$$

for any $\hat{u}$, $\hat{v}$, which immediately yields the functional equation

$$\hat{f}(k+\ell)(\hat{f}(k)+\hat{f}(\ell)) = \hat{f}(k)\hat{f}(\ell)+1 \tag{8}$$

for the function $\hat{f}$. The second condition (7) is simply the requirement that the operator $\mathcal{T}$ should be skew-symmetric. To see this in terms of $\hat{f}$, rewrite (7) as

$$\begin{aligned} 0 &= \mathrm{i}\iiint (\hat{u}(\ell)\hat{v}(k-\ell) + \hat{v}(\ell)\hat{u}(k-\ell))\hat{f}(k-\ell)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k\,\mathrm{d}\ell\,\mathrm{d}x \\ &= 2\pi\mathrm{i}\iint (\hat{u}(\ell)\hat{v}(k-\ell) + \hat{v}(\ell)\hat{u}(k-\ell))\hat{f}(k-\ell)\delta(k)\,\mathrm{d}k\,\mathrm{d}\ell \\ &= 2\pi\mathrm{i}\int \hat{u}(\ell)\hat{v}(-\ell)(\hat{f}(-\ell) + \hat{f}(\ell))\,\mathrm{d}\ell, \end{aligned}$$

whence

$$\hat{f}(\ell) + \hat{f}(-\ell) = 0.$$

Hence the function $\hat{f}$ must be odd.

It is simple to show that the general odd solution of the functional equation (8) is just

$$\hat{f}(k) = \coth(\delta k), \tag{9}$$

where δ is an arbitrary parameter. (The proof is presented in the next section below.) This includes the solution

$$\hat{f}(k) = \mathrm{sgn}(k) \tag{10}$$

as the limiting case $\delta \to \infty$. Solution (9) just gives the Fourier symbol of the coth transform (5), corresponding to the ILW equation (4), while the limiting case (10) produces the Hilbert transform (2) appearing in the Benjamin–Ono equation (1). Hence if the operator $\mathcal{T}$ is constrained to satisfy conditions (6) and (7), then the ILW equation and the Benjamin–Ono equation are the *only* integrable evolution equations of the form (4). These conditions are much more stringent than the requirement that $[\mathcal{T}, \partial_x] = 0$, which is sufficient to ensure Darboux covariance of an associated linear system [10].

In the following section we show how the functional equation (8) may be derived immediately as a necessary condition of integrability within the perturbative symmetry approach. After that we present a short proof that the general odd solution of (8) is given by (9) (see also [4]).

2. Perturbative symmetry approach and the functional equation

Let us consider the equation (4), where the operator $\mathcal{T}$ has the Fourier symbol $\mathrm{i}\hat{f}(k)$, within the framework of the perturbative symmetry approach [11]. After removing the linear term $\delta^{-1}u_x$ from (4) by means of a Galilean transformation, the symbolic representation of the equation reads as

$$u_t = u\omega(k_1) + \frac{u^2}{2}a(k_1, k_2) \equiv F, \tag{11}$$

where

$$\omega(k) = -\mathrm{i}k^2\hat{f}(k), \qquad a(k_1, k_2) = -2(k_1 + k_2). \tag{12}$$

Observe that in (11) the symbol u stands for the Fourier transform, and we write the wavenumber k in place of $\mathrm{i}k$, consistent with the notation in [11]; this means that $\omega(k)$ differs by a factor of i, and is missing a linear term, compared to the physical dispersion relation for water waves [6].

In the perturbative symmetry approach it is supposed that symmetries and local conservation laws of equation (4), if they exist, belong to a proper extension of a differential ring. To construct this, consider a ring $\mathcal{R}[u]$ of differential polynomials over $\mathbb{C}$, generated by u and its x-derivatives and take the sequence of ring extensions

$$\mathcal{R}_{\mathcal{T}}^0 = \mathcal{R}[u], \mathcal{R}_{\mathcal{T}}^1 = \overline{\mathcal{R}_{\mathcal{T}}^0 \bigcup \mathcal{T}\left(R_{\mathcal{T}}^0\right)}, \ldots, \mathcal{R}_{\mathcal{T}}^{n+1} = \overline{\mathcal{R}_{\mathcal{T}}^n \bigcup \mathcal{T}\left(R_{\mathcal{T}}^n\right)}, \ldots, \tag{13}$$

where the set $\mathcal{T}\left(R_{\mathcal{T}}^n\right) = \left\{\mathcal{T}(a), a \in R_{\mathcal{T}}^n\right\}$ and the horizontal line denotes ring closure. Every set $\mathcal{R}_{\mathcal{T}}^n$ is a ring and n indicates the nesting depth of the operator $\mathcal{T}$. The symbolic representation of the extended ring may be constructed as follows. Suppose a is an element of the ring $\mathcal{R}[u]$, of degree m, and its symbolic representation is $a \to u^m a(k_1, \ldots, k_n)$. Then to $\mathcal{T}(a)$ corresponds a symbol $\mathrm{i}u^m \hat{f}(k_1 + \cdots + k_n) a(k_1, \ldots, k_n)$.

The main theorem of the perturbative symmetry approach states that if equation (4) possesses an infinite hierarchy of higher symmetries from the extended ring then there exists a formal recursion operator Λ, with symbol

$$\Lambda = p + u\phi_1(k_1, p) + u^2\phi_2(k_1, k_2, p) + u^3\phi_3(k_1, k_2, k_3, p) + \cdots, \tag{14}$$

which satisfies the equation

$$\Lambda_t = [F_*, \Lambda], \tag{15}$$

where F_* denotes the Fréchet derivative of the right-hand side of (11). Moreover, all the functions $\phi_n(k_1, \ldots, k_n, p)$ must be quasilocal, i.e. the coefficients of their asymptotic expansion in p as $p \to \infty$ are of the form

$$\phi_m(k_1, \ldots, k_m, p) = \sum_{l \leqslant s} \phi_{ml}(k_1, \ldots, k_m) p^l, \tag{16}$$

where each $u^m \phi_{ml}(k_1, \ldots, k_m)$ must belong to the symbolic representation of the extended ring. This statement suggests necessary integrability conditions for the equation (4).

Proposition 1. *Let $\hat{f}(k) \to 1$, faster than any power of k^{-1}, as $k \to +\infty$. Then if the coefficients $\phi_1(k_1, p), \phi_2(k_1, k_2, p)$ of the formal recursion operator (14) for equation (4) are quasilocal, the function $\hat{f}(k)$ must satisfy the functional equation (8).*

Proof. From (15) it follows that functions $\phi_1(k_1, p), \phi_2(k_1, k_2, p)$ are given by the following formulae:

$$\phi_1(k_1, p) = \frac{k_1 a(k_1, p)}{\omega(k_1 + p) - \omega(k_1) - \omega(p)}, \tag{17}$$

$$\phi_2(k_1, k_2, p) = \frac{N_2(k_1, k_2, p)}{\omega(k_1 + k_2 + p) - \omega(k_1) - \omega(k_2) - \omega(p)}, \tag{18}$$

where

$$\begin{aligned} N_2(k_1, k_2, p) = \big[&\tfrac{1}{2}\phi_1(k_1 + k_2, p) a(k_1, k_2) + \langle \phi_1(k_1, p + k_2) a(k_2, p) \\ &- a(k_1, p + k_2)\phi_1(k_2, p)\rangle\big] \end{aligned}$$

with the triangular brackets $\langle , \rangle$ denoting symmetrization over the arguments k_1, k_2. Taking into account (12) we obtain for $\phi_1(k_1, p)$

$$\begin{aligned} \phi_1(k_1, p) &= -\frac{2k_1(k_1 + p)}{\mathrm{i}(k_1 + p)^2 \hat{f}(k_1 + p) - \mathrm{i}k_1^2 \hat{f}(k_1) - \mathrm{i}p^2 \hat{f}(p)} \\ &= \frac{\mathrm{i}(1 + k_1/p)}{1 + \frac{1}{2}k_1(1 - \hat{f}(k_1))/p + O(p^{-\infty})}, \qquad \text{as} \quad p \to +\infty, \end{aligned}$$

where we substituted in the asymptotics $\hat{f}(k_1+p) = 1 + O(p^{-\infty})$, $\hat{f}(p) = 1 + O(p^{-\infty})$. It is clear that $\phi_1(k_1, p)$ is quasilocal. The expansion of $\phi_2(k_1, k_2, p)$ in p when $p \to \infty$ is of the form

$$\phi_2(k_1, k_2, p) = \phi_{2,-2}(k_1, k_2)p^{-2} + \phi_{2,-3}(k_1, k_2)p^{-3} + \phi_{2,-4}(k_1, k_2)p^{-4} + \cdots, \tag{19}$$

where the symbols $u^2\phi_{2,-2}(k_1, k_2)$ and $u^2\phi_{2,-3}(k_1, k_2)$ belong to the symbolic representation of the extended ring, while the coefficient $\phi_{2,-4}(k_1, k_2)$ has the form

$$\phi_{2,-4}(k_1, k_2) = \frac{g(k_1, k_2)}{k_1 + k_2}, \tag{20}$$

with $u^2 g(k_1, k_2)$ being the symbol of an element of the extended ring. (We do not present the explicit expressions here since they are quite large.) Due to the denominator $k_1 + k_2$, in general the function $\phi_{2,-4}(k_1, k_2)$ will *not* correspond to an element of the extended ring and is the obstacle to integrability. To overcome this obstacle, $k_1 + k_2$ must divide the function $g(k_1, k_2)$ (considered as a polynomial in k_1, k_2 with coefficients in terms of $\hat{f}$), and the divisibility conditions are given precisely by (8) with $k = k_1$ and $\ell = k_2$. The proposition is proved. □

Remark. It is also possible to prove the same result under the slightly more general assumption that $\hat{f}(k) \approx 1 + \sum_{j=1}^{\infty} c_j k^{-j}$ as $k \to +\infty$. After applying a Galilean transformation to the equation (4), the term c_1/k can be removed, and then the conditions on $\phi_{2,-4}$ are the same as the above.

Proposition 2. *The most general odd solution to (8) that is analytic everywhere apart from the origin is given by (9) or (10). The only even solution is $\hat{f}(k) = 1$ (constant).*

Proof. To solve the functional equation (8) when $\hat{f}$ is *odd*, set $\ell = -k + h$, and take the limit $h \to 0$ to obtain

$$\lim_{h\to 0} \hat{f}(h)(\hat{f}(k) + \hat{f}(-k+h)) = 1 - \hat{f}(k)^2. \tag{21}$$

Thus if the upper/lower limits $\lim_{h\to 0\pm} \hat{f}(h)$ are both finite then the left-hand side of (21) is zero, which gives $\hat{f}(k) = \pm 1$ for all k, and then (up to an overall sign) the solution is given by (10). On the other hand, if $\hat{f}(h)$ becomes unbounded as $h \to 0$, then (assuming analyticity) the limit in (21) gives

$$\delta^{-1}\hat{f}'(k) = 1 - \hat{f}(k)^2 \qquad \text{with} \quad \delta = (\lim_{h\to 0} h\hat{f}(h))^{-1}. \tag{22}$$

In that case the unique odd solution of the differential equation (22), having the required simple pole at the origin, is precisely (9). If we consider the case when $\hat{f}$ is *even* instead, then setting $\ell = \pm k$ in turn yields

$$2\hat{f}(2k)\hat{f}(k) = 1 + \hat{f}(k)^2 = 2\hat{f}(0)\hat{f}(k)$$

whence $\hat{f}(2k) = \hat{f}(0)$, constant, and it immediately follows that $\hat{f}(k) = 1$ for all k. □

Remark. An alternative proof is furnished by the substitution $\hat{f}(k) = (1 + E(k))/(1 - E(k))$, leading to $E(k)$ satisfying the functional equation for the exponential. This implies that there are no solutions of (8) with indefinite parity. However, the solution (10) does not arise so naturally via this method.

Hence skew-symmetric operators $\mathcal{T}$, satisfying (6) and (7), yield integrable equations of the form (4) as isolated by the perturbative symmetry approach. Moreover, the Benjamin–Ono and ILW equations are the only such equations having infinitely many conservation laws (and solvable by the inverse scattering transform). We should remark that with the even solution

$\hat{f} \equiv 1$ the equation (4) is (up to scaling) just Burgers' equation, which has no non-trivial conservation laws, but nevertheless has infinitely many symmetries and is integrable by direct linearization (i.e. the Hopf–Cole transformation [7, 8]). It is also interesting to note that, after rescaling $\hat{f}$ by suitable powers of δ, the shallow water limit $\delta \to 0$ can be made. Taking the limit directly yields $\hat{f}(k) \sim 1/(\delta k)$, which means that (4) becomes the Riemann shock equation; this is the dispersionless limit of the KdV equation, for which $\hat{f}(k) \sim \delta k$ arises by first transforming $\hat{f}(k) \to \hat{f}(k) - 1/(\delta k)$ before taking $\delta \to 0$.

3. Periodic case

The purpose of this section is to describe how the symbolic method and the perturbative symmetry approach can be carried over to the case of periodic functions with period $2L$. For such a function $u(x)$, by considering the Fourier series

$$u(x) = \sum_{n=-\infty}^{\infty} u_n \, \mathrm{e}^{\mathrm{i}n\pi x/L},$$

the Fourier transform is obtained as a sum of Dirac delta functions:

$$\hat{u}(k) = \sum_{n=-\infty}^{\infty} u_n \delta(n\pi/L - k).$$

Thus, applying the operator $\mathcal{T}$ with symbol $\mathrm{i}\hat{f}(k)$ leads to

$$\mathcal{T}(u(x)) = \mathrm{i}\int_{-\infty}^{\infty} \hat{f}(k) \sum_{n=-\infty}^{\infty} u_n \delta(n\pi/L - k)\, \mathrm{e}^{\mathrm{i}kx}\, \mathrm{d}k = \mathrm{i}\sum_{n}{}' \hat{f}(n\pi/L) u_n \, \mathrm{e}^{\mathrm{i}n\pi x/L},$$

where the symbol $\sum_n'$ denotes the sum from $n = -\infty$ to ∞ with $n = 0$ excluded in the case of a principal value integral. Hence the perturbative symmetry approach is applicable in a periodic domain, by formally replacing u by its Fourier coefficient u_n, and similarly letting $\hat{f}(n\pi/L)$ for $n \in \mathbb{Z}$ take the place of the symbol $\hat{f}(k)$; the results on the infinite domain are recovered in the limit $L \to \infty$. However, from an algebraic point of view the integrability conditions are identical.

To consider the periodic problem without recourse to distributions, it is convenient to follow [1] and represent the operator $\mathcal{T}$ in terms of an integral kernel on the interval $-L \leqslant x \leqslant L$:

$$\mathcal{T}(u(x)) = \frac{1}{2L} PV \int_{-L}^{L} \tilde{\mathcal{T}}(x - y) u(y)\, \mathrm{d}y. \tag{23}$$

From the result of applying the perturbative symmetry approach in the last section, it is clear that in the periodic case an integrable equation (4) with a skew-symmetric operator $\mathcal{T}$ can *only* arise from an odd solution of the functional equation (8). In that case the kernel must be given by the Fourier series

$$\tilde{\mathcal{T}}(x) = \mathrm{i}\sum_{n}{}' \coth(n\pi\delta/L)\, \mathrm{e}^{\mathrm{i}n\pi x/L} \tag{24}$$

or its limit when $\delta \to \infty$. The authors of [1] used this Fourier series to obtain a closed form expression for the periodic coth transform on $(-L, L)$, which they identified as

$$\tilde{\mathcal{T}}(x) = -\frac{2K}{\pi}\{Z(Kx/L) + \mathrm{dn}(Kx/L)\mathrm{cs}(Kx/L)\}, \tag{25}$$

with Z denoting Jacobi's zeta function and dn, cs being Jacobian elliptic functions; the constants K and K' are complete elliptic integrals such that $K'/K = \delta/L$.

Here we should point out that the sum of the series (24) can be written more succinctly. By using the identities

$$Z(w) = \frac{\mathrm{d}}{\mathrm{d}w} \log \vartheta_4\big(w\vartheta_3^{-2}\big), \qquad \mathrm{dn}(w)\mathrm{cs}(w) = \frac{\mathrm{d}}{\mathrm{d}w} \log \frac{\vartheta_1\big(w\vartheta_3^{-2}\big)}{\vartheta_4\big(w\vartheta_3^{-2}\big)}$$

(see [18]), it is clear that the expression (25) can be simplified to rewrite (23) as

$$\mathcal{T}(u(x)) = -\frac{1}{\pi} PV \int_{-L}^{L} u(y) \frac{\mathrm{d}}{\mathrm{d}x} \log \vartheta_1((x-y)\pi/2L)\,\mathrm{d}y,$$

using the fact that $\vartheta_3^2 = 2K/\pi$. Alternatively the kernel $\tilde{\mathcal{T}}(x)$ can be written as a zeta function $Z\big[{1 \atop 1}\big]$ with characteristics [15].

In future work we propose to consider elliptic solutions of the periodic ILW equation of the form

$$u(x,t) = \sum_{j=1}^{N} F(x - q_j(t)) - F(x - \overline{q}_j(t)),$$

where F is a suitable zeta function. The simplest case $N = 1$ corresponds to the elliptic solutions studied by Parker [14] using Hirota's bilinear method. For arbitrary N the quantities $q_j, \overline{q}_j$ should evolve with time according to the equations of an elliptic Calogero–Moser system coupled with first-order constraints. This generalizes the results of Case [5], which relate the algebraic solitons of the Benjamin–Ono equation to rational Calogero–Moser systems. There are various interesting integrable equations involving the Hilbert transform operator $\mathcal{H}$ that have been classified recently [12], and it would be instructive to consider their solutions and their analogues when $\mathcal{H}$ is replaced by $\mathcal{T}$.

Acknowledgments

ANWH would like to thank Harry Braden for useful discussions, and is grateful to the University of Kent for a Colyer-Fergusson award for the project *Algebraic curves and functional equations in mathematical physics*. VSN is funded by a Royal Society NATO/Chevening fellowship on the project *Symmetries and coherent structures in non-evolutionary partial differential equations*. We are both grateful to Sasha Mikhailov for many interesting discussions on related matters.

References

[1] Ablowitz M J, Fokas A S, Satsuma J and Segur H 1982 *J. Phys. A: Math. Gen.* **15** 781
[2] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia, PA: SIAM)
[3] Benjamin T B 1967 *J. Fluid Mech.* **29** 559
[4] Braden H W and Byatt-Smith J G 2003 *SIAM J. Math. Anal.* **34** 736
[5] Case K M 1978 *J. Math. Phys.* **20** 972
[6] Chen H H and Lee Y C 1979 *Phys. Rev. Lett.* **43** 264
[7] Cole J D 1951 *Q. Appl. Math.* **9** 225
[8] Hopf E 1950 *Commun. Pure Appl. Math.* **3** 201
[9] Joseph R J 1977 *J. Phys. A: Math. Gen.* **10** L225
[10] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[11] Mikhailov A V and Novikov V S 2002 *J. Phys. A: Math. Gen.* **35** 4775

[12] Mikhailov A V and Novikov V S 2003 *Mos. Math. J.* **3** 1293

[13] Ono H 1975 *J. Phys. Soc. Japan.* **39** 1082

[14] Parker A 1992 *J. Phys. A: Math. Gen.* **25** 2005

[15] Rauch H E and Lebowitz A 1973 *Elliptic Functions, Theta Functions, and Riemann Surfaces* (Baltimore, MD: Williams & Wilkins)

[16] Sanders J and Jing Ping Wang 1998 *J. Differ. Equ.* **147** 410

[17] Satsuma J, Ablowitz M J and Kodama Y 1979 *Phys. Lett.* A **73** 283

[18] Whittaker E T and Watson G N 1965 *A Course of Modern Analysis* 4th edn (Cambridge: Cambridge University Press)